Quiz 4 Solutions

1. Find the centroid of the region bounded by $y = \sin(x)$ and the x-axis between $x = 0$ and $x = \pi$.
Solution: $\overline{x} = \frac{\pi}{2}$ by symmetry of $\sin x$ about $x = \frac{\pi}{2}$. We have $A = \int_0^\pi \sin x \, \mathrm{d}x = -\cos x\Big|_0^\pi = 2$ and

$$\begin{aligned}\overline{y} &= \frac{1}{A}\int_0^\pi \frac{1}{2}\sin^2(x)\mathrm{d}x = \frac{1}{8}\int_0^\pi 1-\cos(2x)\mathrm{d}x = \frac{1}{8}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^\pi \\ &= \frac{\pi}{8}.\end{aligned}$$

So the centroid is $(\frac{\pi}{2}, \frac{\pi}{8})$.
Instead of arguing $\overline{x} = \frac{\pi}{2}$ by symmetry one can of course also compute

$$\begin{aligned}\overline{x} &= \frac{1}{A}\int_0^\pi x\sin(x)\mathrm{d}x = \frac{1}{2}\left(-x\cos x\Big|_0^\pi + \int_0^\pi \cos x \, \mathrm{d}x\right) \\ &= \frac{1}{2}\left(\pi + (\sin x\Big|_0^\pi)\right) \\ &= \frac{\pi}{2}.\end{aligned}$$

2. Determine whether the sequence given by $a_n = \frac{\ln(n)}{\ln(2n)}$ converges or diverges. If it converges, find the limit.
(As always you should fully justify your answer)
Solution:

$$a_n = \frac{\ln n}{\ln 2 + \ln n} = \frac{1}{\underbrace{\frac{\ln 2}{\ln n}}_{\to 0} + 1} \to 1$$

So the sequence converges and has limit 1.
Alternatively, apply L'Hôpital's rule:

$$\lim_{n\to\infty}\frac{\ln n}{\ln(2n)} = \lim_{x\to\infty}\frac{\ln x}{\ln(2x)} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{2}{2x}} = \lim_{x\to\infty} 1 = 1.$$

3. For which real numbers x does the series $\sum_{n=0}^{\infty}(e^x)^n$ converge? For those x determine the value of the sum in closed form.

Solution: This is a geometric series with common ratio $r = e^x$. So it converges iff $|r| < 1$, or equivalently if $x < 0$. In this case the sum is $\frac{1}{1-r} = \frac{1}{1-e^x}$.